Maximization and Minimization with Inequalities Without Using Kuhn-Tucker

Calculate the absolute extrema of $f(x, y) = x^2 + 2y^2$ over the region $C = \{(x, y) \in \mathbb{R}/x^2 + y^2 \leq 1\}$

Solution

One way to approach this problem is by using the Kuhn-Tucker method for constrained maximization or minimization problems with inequalities. However, analyzing the form of the function, we can see that if we want to maximize, we know that the constraint will be fulfilled with equality since there is no reason to choose a smaller x or y. If we want to maximize, then x and y must be as high as possible.

Considering Maximums

To calculate the maximum, we can take $x^2 + y^2 = 1$. The Lagrangian will then be:

$$L = x^2 + 2y^2 + \lambda(1 - x^2 - y^2)$$

The first-order conditions:

$$L'x = 2x - \lambda 2x = 0$$
$$L'y = 4y - \lambda 2y = 0$$
$$L'\lambda = 1 - x^2 - y^2 = 0$$

Rewriting the first two:

$$L'x = 2x(1 - \lambda) = 0$$
$$L'y = 2y(2 - \lambda) = 0$$

Assuming $x \neq 0$, from the first equation we can solve for λ:

$$2x = \lambda 2x$$

$$\lambda = 1$$

Then in the second equation:

$$2y = y$$

This only holds if $y = 0$. And this with the constraint $(1 - x^2 - y^2 = 0)$ implies that $x = 1$ or $x = -1$. This leaves us with two points: $(1, 0)$ and $(-1, 0)$. Let's consider another path, assuming $x = 0$. In the third equation:

$$1 - 0 - y^2 = 0$$

We get: $y = 1$ and $y = -1$. Therefore, we have two critical points: $(0, 1)$ and $(0, -1)$. This gives us a value of $\lambda = 2$, for the second condition to hold.
In summary, we have:

- $f(0, 1) = 2$
- $f(0, -1) = 2$
- $f(1, 0) = 1$
- $f(-1, 0) = 1$

Remember, we are looking to maximize, so we discard the last two points since they give a value of 1, which is less than 2. We perform the second-order conditions:

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2 - 2\lambda & 0 \\ 2y & 0 & 4 - 2\lambda \end{pmatrix}$$

We obtain the determinant:

$$-2x \begin{vmatrix} 2x & 2y \\ 0 & 4 - 2\lambda \end{vmatrix} + 2y \begin{vmatrix} 2x & 2y \\ 2 - 2\lambda & 0 \end{vmatrix} = -2x(8x - 4x\lambda) + 2y(-4y + 4y\lambda) = -8x^2(2 - \lambda) - 8y^2(-1 + \lambda)$$

Remember that if $x = 0$ and $y = 1$, then $\lambda = 2$. We evaluate at the points:

$$0 - 8 * (-1 + 2) = 8 > 0$$

$$0 - 8 * (-1 + 2) = 8 > 0$$

We have two maximums.

Considering Minimums

We know that both x^2 and y^2 enter the function positively, so if we want to minimize the function, we would want those two numbers to be as small as possible. Therefore, the constraint would be $x^2 + y^2 = 0$ since $x^2 + y^2$ can never take negative values. At the same time, as the objective function is $x^2 + 2y^2$, we can see that if $x = 0$ and $y = 0$, the function takes a minimum at $(0, 0, 0)$. We can check this analytically if we work with the function without any constraints:

$$f'x = 2x = 0$$

$$f'y = 4y = 0$$

For the first-order conditions to hold, $x = 0$ and $y = 0$. We form the Hessian with the second derivatives:

$$f''_{xx} = 2$$

$$f''_{yy} = 4$$

$$f''_{xy} = f''_{yx} = 0$$

$$|H| = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 > 0$$

Also, as $f''_{xx} = 2 > 0$, we have a minimum.